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Resonance instabilities in the boundary-layer flow over a rotating disk under the influence of a uniform magnetic field

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Abstract Instabilities due to resonating waves in the three-dimensional incompressible viscous/inviscid rotating-disk boundary-layer flow are investigated numerically. The influence of a normal magnetic field on the resonances leading, respectively, to the direct spatial instability, the direct temporal instability and the absolute instability are worked out, in an attempt to determine the physically most significant one. It is found that the magnetic field has a stabilizing influence on all the resonance mechanisms outlined. However, the direct spatial-resonance-instability mechanism persists for low Reynolds numbers, when the flow is still in the laminar regime. Thus, attention should be directed to this specific instability mechanism, which offers a strong route to transition to turbulence by means of triggering the nonlinearity.

Keywords Absolute instability · Conducting fluid · Direct spatial resonance · Direct temporal resonance · Rotating-disk flow

1 Introduction

Resonance-instability mechanisms are of great interest in recent research, due mainly to their capability of breaking the laminar flow and leading to transition to turbulence. They occur in several practical hydrodynamic and hydromagnetic fluid-flow problems involving rotating disks, for instance rotating machinery, rotating stars, turbine blades, computer storage devices and electromagnetic stirring of liquid metals. Such mechanisms have both linear and nonlinear significance and appear to have potential implications for many stability problems. The current work particularly concentrates on the investigation of the resonance instability mechanisms existing in three-dimensional viscous or inviscid incompressible conducting fluid flow. When the fluid is non-conducting, the boundary layer due to a rotating disk is also known as the Von Kármán self-similar solution. Although the rotating disk constitutes a prototype for many boundary-layer flows, our consideration here is in connection with the fully three-dimensional boundary-layer flow over aerodynamic vehicles, such as a swept-wing. The significance lies in the fact that the instabilities existing on both configurations are known to exhibit close similarities. With this purpose in mind, we study the direct spatial resonance, the direct temporal resonance and the absolute instability mechanisms, and their

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properties to contribute towards the final breakdown of the laminar flow here. A uniform magnetic field is assumed to affect the boundary-layer flow in the wall-normal direction.

When a boundary-layer flow is exposed to a line forcing, the space-time evolution of a linear instability is described by a Green's function (see Sect. 4.2 for a description). As demonstrated in [1,2], the local response of the instability field can be measured asymptotically with the help of resonating points, at which the evolution of eigenmode spectrum is able to yield the coalescence of two or more distinct or similar modes. The resonance mechanism that results from the coalescence of two eigenmodes having zero group velocity and also initially originating from the distinct wavenumber planes in the instability field is known as the absolute instability. This mechanism is well-documented in the literature for a non-conducting Von Kármán flow; see for instance [3–5] amongst many others. Moreover, the recent work of Jasmine and Gajjar [6] concluded that the presence of a normal magnetic field acts in the way of stabilizing the absolute instability mechanisms.

However, there can be more dangerous resonance mechanisms generated as a result of the coalescence of two eigenmodes originally emerging from the same wavenumber planes, which are the main subject of the present paper. In this case the local response of the flow is evaluated at a double pole of the Green's function (see Sect. 4.2), yielding either a direct spatial resonance in a coordinate axis or a direct temporal resonance in time. For the examples of the temporal or spatial responses of some flows and their nonlinear consequences, one may refer to [7,8]. It was also thoroughly demonstrated by Turkyilmazoglu and Gajjar [9] that the algebraic growth of the perturbations in a Von Kármán flow starts at a Reynolds number of 445, prior to the appearance of the local absolute instability, and while the flow is still in the laminar state. Therefore, it was conjectured that the first appearance of the nonlinearity may be attributed to the direct spatial resonance instability mechanism rather than the mechanism of absolute instability. Indeed, a recent numerical simulation of Davies and Carpenter [10], using the complete linearized Navier–Stokes equations, clearly indicates that the long-term behavior of the flow appears to be dominated by the convective instability in the real spatially inhomogeneous flow. Thus, reference [10] concludes that the absolute instability does not lead to a sustained temporal growth, which is also supported by the most recent experimental study carried out by Othman [11]. As a result, the direct resonance mechanisms seem to outweigh the absolute instability mechanism investigated by the aforementioned researchers.

In the current paper we consider the rotating-disk boundary-layer flow in an electrically conducting fluid, in the presence of a magnetic field applied in the direction normal to the disk surface, similar to the work of Jasmine and Gajjar [6]. To derive a self-similar form for the mean flow, we make use of the assumptions that the effects of the electric field are of no significance, and that the fluid motion can not alter the applied magnetic field, being constant everywhere. The only effect of the magnetic field on the boundary-layer flow is the presence of the Lorentz force which appears in the momentum equations as an extra forcing term. It is further assumed that the magnetic field is strongly influenced by diffusive effects because of a low-magnetic-field Reynolds number. Under these conditions, our main interest lies here in the determination of possible resonance instabilities that are not only restricted to the absolute or convective types as investigated in [6], but include the direct resonance instability mechanisms due to the spatial or temporal development of the modes on which the present study focusses. As discussed above, the absolute instability results of Jasmine and Gajjar [6] may not be the most relevant from a physics point of view, although their availability serves to validate our numerical scheme as well as justify our numerical findings. The primary goal is hence to work out the most sensitive resonance mechanism in terms of physical importance.

Direct resonances are potentially very dangerous interaction mechanisms. The effective time scales or spatial extents are much shorter than in the self-interactions or resonant triad approaches. In addition to this, relatively large amplitudes result (see Sect. 4.2), so that the three-dimensional disturbances will rapidly disturb the main flow field. Thus, a search for the degenerating eigenmodes and their responses are the objectives of our study. Particularly, the conclusion of the investigations made by Davies and Carpenter [10] and Othman [11] has strongly motivated us to look for branch points leading to direct spatial/

temporal resonances in viscous rotating-disk boundary-layer flow and their inviscid orientation, if it exists. If the damping rates are fairly small, growth is very slow, and, therefore, a direct resonance can give rise to amplitudes increasing much faster than does an exponentially growing mode. Although these modes ultimately decay according to linear theory, the locally high amplitudes may initiate a nonlinear state. In order to determine whether this is possible for the flow under consideration, we also aim to look for the conditions at which a direct resonance exists, by locating damping rates as small as desired.

Our ultimate goal is to find the critical Reynolds numbers for the onset of the resonance instability mechanisms, which is physically most relevant to the flow considered here. The results, in fact, clearly point to the stabilizing influence of the normal magnetic field, via increasing the critical values of the Reynolds numbers for all the resonance mechanisms. The most dangerous mechanism, however, seems to be due to the direct spatial resonance, as compared to the temporal resonance or the absolute instability mechanisms.

This paper is organized as follows. Section 2 presents the governing equations of the viscous incompressible boundary-layer flow over a rotating disk under the influence of a uniform normal magnetic field, as well as the basic generalized Von Kármán flow. The stability equations obtained from linearizing the full viscous incompressible equations around the mean conducting fluid flow are given in Sect. 3, together with the inviscid Rayleigh equation in the large-Reynolds-number limit. Numerical results of the basic flow are then briefly outlined in Sect. 4, followed by a discussion on the resonance instability mechanisms. Section 5 contains the conclusions made from the results.

2 Formulation of the problem

2.1 Governing equations of the flow

We are concerned here with the three-dimensional, unsteady flow of an incompressible, electrically conducting viscous fluid over an infinite disk rotating with a constant angular velocity Ω about its axis of rotation, z. We also assume that in the direction normal to the disk surface a uniform magnetic field $B = B_0$ is applied. The Navier–Stokes equations are non-dimensionalized with respect to a length scale $L = r_e^*$, velocity scale $U_c = L\Omega$, time scale L/U_c and pressure scale ρU_c^2, where ρ is the fluid density and the magnetic field $\mathbf{B}$ with B_0. Such a dimensionless analysis leads to a global Reynolds number $\mathrm{Re} = U_c L/\nu = R^2$, where R is the Reynolds number based on the displacement thickness $\delta = (\frac{\nu}{\Omega})^{\frac{1}{2}}$. Thus, relative to non-dimensional cylindrical polar coordinates (r, θ, z) which rotate with the disk, the full time-dependent, unsteady Navier–Stokes equations governing the viscous magnetohydrodynamic fluid flow are given by

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2(\hat{k} \times \mathbf{u}) - \mathrm{r}\hat{r} = -\nabla p + m\mathbf{J} \times \mathbf{B} + \frac{1}{R^2}\nabla^2 \mathbf{u}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{R_m}\nabla^2 \mathbf{B}, \tag{3}$$

$$\nabla \cdot \mathbf{J} = 0, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{6}$$

In this analysis the fluid is assumed to lie in the semi-infinite space $z \geq 0$. In the above equations ∇^2 is the usual Laplace operator in cylindrical coordinates; R_m defines the magnetic Reynolds number given by

$R_m = \sigma\mu l^2\Omega$, with σ the electrical conductivity of the fluid, μ the free-space magnetic permeability, and also m the magnetic interaction parameter which is positive and defined by $m = \sigma B_0^2/(\rho\Omega)$. In addition to this, the current density $\mathbf{J}$ is given by $\mathbf{J} = \mathbf{E} + \mathbf{u} \times \mathbf{B}$, with E being the electric field. Moreover, the magnetic Reynolds number is assumed to be much smaller than the Reynolds number of the fluid, which is the case for several practical situations. Thus, the applied magnetic field is unaffected by the effect of the motion of the conducting fluid, due to a low magnetic Reynolds number. The effect of the magnetic field on the fluid motion manifests itself in the form $m\mathbf{J} \times \mathbf{B}$ on the right-hand side of the momentum equations (2). This is known as the Lorentz force, the components of which can be evaluated as $(-u, -v, 0)$ for a normal electric field. It is further assumed that there are no radial or azimuthal currents influencing the motion of the fluid flow under consideration.

2.2 Mean flow

The dimensionless mean flow velocities and pressure are given by Von Kármán's exact self-similar solution of the Navier–Stokes equations for steady laminar flow in the case of a non-conducting fluid. From now on the Reynolds number is assumed to be large. Because the boundary-layer thickness is of order R^{-1}, the steady incompressible boundary-layer flow in the presence of a uniform transverse magnetic field over a rotating disk evolves along a boundary-layer coordinate of order unity, defined by $Z = Rz$, as for the non-magnetic case. Then the mean flow quantities take the form

$$(u_B, v_B, w_B, p_B) = \left(rU[Z], rV[Z], \frac{1}{R}W[Z], \frac{1}{R^2}P[Z]\right), \tag{7}$$

where the functions U, V, W and P satisfy the following ordinary differential equations

$$\begin{aligned} &U^2 - (V+1)^2 + U'W - U'' + mU = 0,\\ &2U(V+1) + V'W - V'' + m(V+1) = 0,\\ &P' + W'W - W'' = 0,\\ &2U + W' = 0. \end{aligned} \tag{8}$$

Here, a prime denotes a derivative with respect to Z and the appropriate boundary conditions are given as

$$\begin{aligned} &U = V = W = 0 \quad \text{at } Z = 0,\\ &U = 0,\ V = -1,\ W = h_\infty \quad \text{as } \ Z \to \infty. \end{aligned} \tag{9}$$

The unknown h_∞ is a constant vertical velocity of the rotating and electrically conducting fluid in the far-field above the disk, and its value has to be found numerically in the course of the solution of Eqs. 8–9. It is clear that Eqs. 8–9 reduce to the usual Von Kármán equations in the non-magnetic case, as used extensively in the literature; see for instance Malik [12]. The mean-flow equations of the conducting fluid (8–9) were also used in the context of the finding similarity solutions under the influence of the magnetic resonance parameter m for a rotating-disk boundary-layer flow; see for example Sparrow [13].

3 Linear stability equations

In the present work we are interested in perturbation solutions of Von Kármán's self-similarity velocity profiles (8–9). The instantaneous non-dimensionalized velocity components imposed on the basic steady flow are u, v, w and the pressure component is p and they can be expressed as

$$(u, v, w, p) = (u_B, v_B, w_B, p_B) + (u', v', w', p').$$

Therefore, infinitesimally small disturbances u', v', w' and p' are superimposed on the steady flow obtained from the solution of (8–9). The disturbance components of the above system are determined later by solving the form of the Navier–Stokes equations that results from substituting these quantities in (1–6), and subtracting out the mean flow equations, satisfying (8–9). Having linearized the equations for small perturbations, we find that the linearized Navier–Stokes operator has coefficients independent of θ, and hence, the disturbances can be decomposed into a normal mode form proportional to $\mathrm{e}^{\mathrm{i}R(\beta\theta-\bar{\omega}t)}$. Such an approximation causes the disturbances to be wave-like and separable in θ and t. Consequently, the perturbations may be assumed to be of the form

$$(u', v', w', p') = (\tilde{u}[r,Z], \tilde{v}[r,Z], \tilde{w}[r,Z], \tilde{p}[r,Z])\mathrm{e}^{\mathrm{i}R(\beta\theta-\bar{\omega}t)} + c.c.,$$

where β and $\bar{\omega}$ are, respectively the wave number in the azimuthal direction and the scaled frequency of the wave propagating in the disturbance-wave direction.

Separation in θ and t simplifies the linear system of equations. However, no such simplification arises as far as the r-dependence is concerned (except in the limit as $R \to \infty$), and the full linearized partial differential system has to be solved subject to suitable initial conditions to determine the stability of the flow. Consider next the Reynolds number of the flow to be sufficiently large, and hence introduce the scale $X = Rr$, which is the appropriate scale for the development of disturbances. After allowing for the multiple-scale replacement of $\partial/\partial r$ by

$$R\frac{\partial}{\partial X} + \frac{\partial}{\partial r},$$

and keeping only terms of up to O($1/R$), the following linear system is obtained

$$\begin{aligned}
&-\mathrm{i}\bar{\omega}\tilde{u} + rU\frac{\partial \tilde{u}}{\partial X} + \mathrm{i}\beta V\tilde{u} + r\frac{\mathrm{d}U}{\mathrm{d}Z}\tilde{w} + \frac{\partial \tilde{p}}{\partial X} \\
&\quad = -\frac{1}{R}\left[W\frac{\partial \tilde{u}}{\partial Z} + U\tilde{u} - 2(V+1)\tilde{v} - \nabla_2^2\tilde{u}\right] - m\tilde{u} - \frac{1}{R}\left[rU\frac{\partial \tilde{u}}{\partial r} + \frac{\partial \tilde{p}}{\partial r}\right],
\end{aligned}$$

$$\begin{aligned}
&-\mathrm{i}\bar{\omega}\tilde{v} + rU\frac{\partial \tilde{v}}{\partial X} + \mathrm{i}\beta V\tilde{v} + r\frac{\mathrm{d}V}{\mathrm{d}Z}\tilde{w} + \mathrm{i}\frac{\beta}{r}\tilde{p} \\
&\quad = -\frac{1}{R}\left[W\frac{\partial \tilde{v}}{\partial Z} + U\tilde{v} + 2(V+1)\tilde{u} - \nabla_2^2\tilde{v}\right] - m\tilde{v} - \frac{1}{R}\left[rU\frac{\partial \tilde{v}}{\partial r}\right],
\end{aligned}$$

$$-\mathrm{i}\bar{\omega}\tilde{w} + rU\frac{\partial \tilde{w}}{\partial X} + \mathrm{i}\beta V\tilde{w} + \frac{\partial \tilde{p}}{\partial Z} + \frac{1}{R}\left[W\frac{\partial \tilde{w}}{\partial Z} + \frac{\mathrm{d}W}{\mathrm{d}Z}\tilde{w} - \nabla_2^2\tilde{w}\right] = -\frac{1}{R}\left[rU\frac{\partial \tilde{w}}{\partial r}\right],$$

$$\frac{\partial \tilde{u}}{\partial X} + \frac{1}{Rr}\tilde{u} + \frac{\mathrm{i}\beta}{r}\tilde{v} + \frac{\partial \tilde{w}}{\partial Z} = -\frac{1}{R}\frac{\partial \tilde{u}}{\partial r}. \tag{10}$$

The operator ∇_2^2 is defined by

$$\nabla_2^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \frac{\beta^2}{r^2}.$$

The terms on the right-hand side of (10) reflect the non-parallelism of the basic flow, and appear at the same order as the other O($1/R$) terms, which are retained in the familiar 'parallel flow approximation'. In the formal limit $R \to \infty$, and with $\frac{\partial}{\partial X}$ replaced by iα, we obtain Rayleigh's equation. It is only in this limit that the full normal-mode decomposition can be justified. If we neglect the terms on the right-hand side of (10) and replace $\frac{\partial}{\partial X}$ by iα together with $r = 1$, the sixth-order system similar to the non-magnetic linear stability equations studied by several people including Malik [12] and Lingwood [3], amongst others, is retrieved, which was recently derived for the conducting-fluid case by Jasmine and Gajjar [6]. Setting $r = 1$, as highlighted above, is tantamount to considering the stability at the local dimensional radial station r_e^*.

The reduced system of equations stemming from these approximations can be written in the following form

$$\begin{aligned}
&\tilde{u}'' - W\tilde{u}' - [\mathrm{i}RU_B + \lambda^2 + U + m]\tilde{u} + 2(V+1)\tilde{v} - RU'\tilde{w} - \mathrm{i}\alpha R\tilde{p} = 0,\\
&\tilde{v}'' - W\tilde{v}' - [\mathrm{i}RU_B + \lambda^2 + U + m]\tilde{v} - 2(V+1)\tilde{u} - RV'\tilde{w} - \mathrm{i}\beta R\tilde{p} = 0,\\
&\tilde{w}'' - W\tilde{w}' - [\mathrm{i}RU_B + \lambda^2 + W']\tilde{w} - R\tilde{p}' = 0,\\
&\bar{\alpha}\tilde{u} + \mathrm{i}\beta\tilde{v} + \tilde{w}' = 0,
\end{aligned} \tag{11}$$

where $U_B = (\alpha U + \beta V - \bar{\omega})$, $\lambda^2 = \alpha^2 + \beta^2$, $\bar{\alpha} = \mathrm{i}\alpha + \frac{1}{R}$ and $\omega = \bar{\omega}R$.

The boundary conditions for this set of equations are $\tilde{u} = \tilde{v} = \tilde{w} = 0$ at the solid wall ($Z = 0$). Considering the property of decaying disturbances, we derive the boundary conditions to be imposed far away from the disk surface from the asymptotic form of Eq. 11, or simply, zero perturbations are imposed. It should be noted here that the viscous nature of the resonance instabilities will be explored by numerically integrating (11) for the associated eigenvalues.

It is also possible to obtain the familiar Orr–Sommerfeld equation for the normal velocity component from the set (11), by ignoring the streamline curvature and Coriolis effects. Further neglect of all the terms of order of R^{-1} in (11) leads to the well-known Rayleigh equation governing the inviscid hydrodynamic stability of the magnetic fluid

$$U_B\tilde{w}'' - (\lambda^2 U_B + U_B'')\tilde{w} = 0. \tag{12}$$

The homogeneous boundary conditions to be incorporated in (12) are also given as

$$\tilde{w} = 0 \;\text{ at }\; Z = 0, \quad \tilde{w} \to 0 \;\text{ as }\; Z \to \infty. \tag{13}$$

The inviscid nature of the resonances will be evaluated from (12) to complement their viscous origin.

4 Results and discussion

Both the mean-flow equations (8–9) as well as the disturbance equations (11–13) were solved making use of a numerical procedure based on Chebyshev polynomials. The details of the integration technique can be found in [14]. In what follows, the numerical results obtained will be discussed.

4.1 Mean flow results

Equations 8–9 were solved first numerically to compute the basic flow profiles of Von Kármán influenced by the presence of a uniform normal magnetic field. Figure 1 demonstrates the mean flow field (U, V, W) displayed for several magnetic field-strength parameters m. It is apparent that the mean flow profiles are radically decreased, and, the boundary-layer thickness is much reduced for increasing magnetic field parameter m. Such an effect of the normal magnetic field enables one to think of the stabilizing impact on the disturbances entering the boundary layer, which will indeed be the case as shown later.

4.2 Resonance results

A flow field governed by the Navier–Stokes equations can sustain in general three types of modal instability wave fields, namely acoustic, vorticity and entropy waves. In the linear approximation these modal fields are uncoupled, except possibly at direct resonance. The instability of the boundary-layer flow over a rigid rotating disk and a consequent transition route to turbulence are almost certain to occur through three distinct families of eigensolutions created by the aforementioned physical waves, and identified so far from the system of equations (11–13). These are, respectively, the inviscid mode of Gregory et al. [15], the

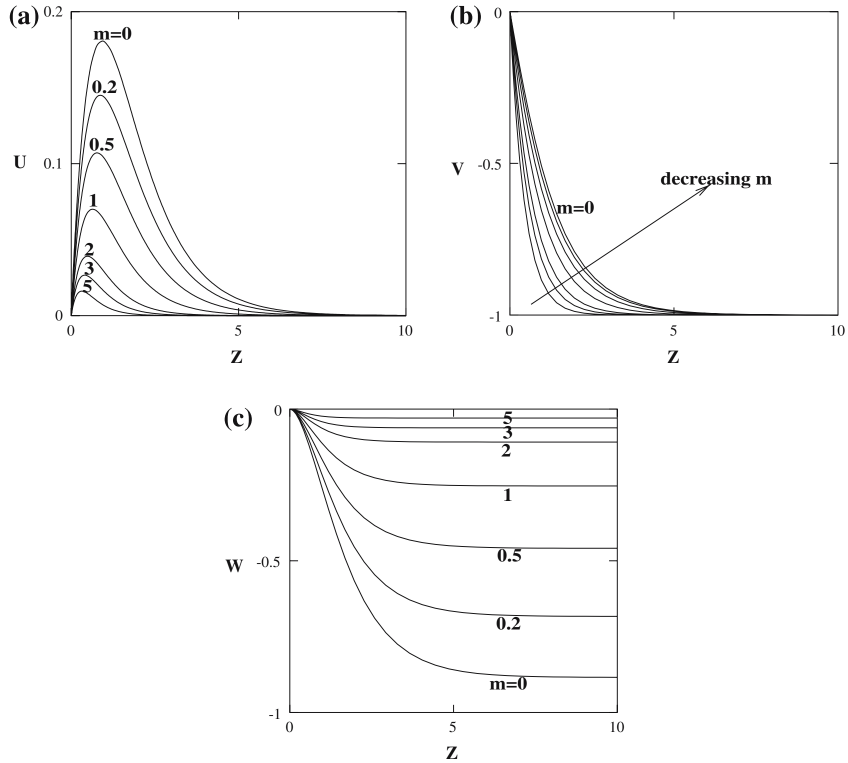

Fig. 1 Basic flow quantities for the rotating-disk flow of conducting fluid are shown for varying magnetic field-strength parameters m, respectively in (**a**) the radial velocity profiles, (**b**) the circumferential velocity profiles and (**c**) the wall-normal velocity profiles

viscous mode balanced by the Coriolis effects and a third family computed first in [16]. When these modes act together and thus constitute a coalescence in the wavenumber plane, they lead to two well-known strong instability mechanisms. One is the local absolute instability which takes place by the process of pinching of inviscid and viscous modes; this subject is investigated in [2,3,14,17,18]. On the other hand, the modal coalescence taking place via the gathering of inviscid mode and the mode of Mack [16] causes a resonance at radial positions or specific times, respectively denoted as, direct spatial resonance, or, direct temporal resonance. To get a close look at the resonance instability mechanisms, we give a brief mathematical foundation as follows. In order to distinguish between the resonance instability mechanisms, the governing equations, whether viscous (11), or inviscid (12), are solved subject to an impulsive azimuthal line forcing with some prescribed integer β, such that the vertical no-slip velocity on the wall is replaced by $\delta(r-r_a)\delta(t)\mathrm{e}^{\mathrm{i}\beta\theta}$, where $\delta(r-r_a)$ and $\delta(t)$ are Dirac delta functions at $r=r_a$ and $t=0$, respectively. The response to point forcing can be obtained by summing over all integer values of β. With such an initial boundary-value perturbation, the inhomogeneous problem reduces to solving a Green's function of the subsequent form for a particular Reynolds number, see for example Lingwood [2],

$$G(r,\theta,z,t) = \frac{1}{2\pi^2}\int_F\int_L \frac{\Psi(\alpha,\beta,z,\omega)}{D(\alpha,\beta,\omega)}\mathrm{e}^{\mathrm{i}[\alpha(r-r_a)+\beta\theta-\omega t]}\mathrm{d}\omega\mathrm{d}\alpha, \tag{14}$$

where, Ψ is a function that consists of eigenvectors of the unforced problem (11) or (12), $D(\alpha,\beta,\omega)=0$ is the dispersion relation, which is satisfied by the discrete eigenvalues of the homogenous problem, and, F and L are inversion contours in the α- and ω-planes, respectively.

Brigg's method in [19, Chapter 2] is later used to estimate the time-asymptotic discrete solution, due to the discrete poles of (14) (that is the zeros of the dispersion relation). If the poles are simple, a

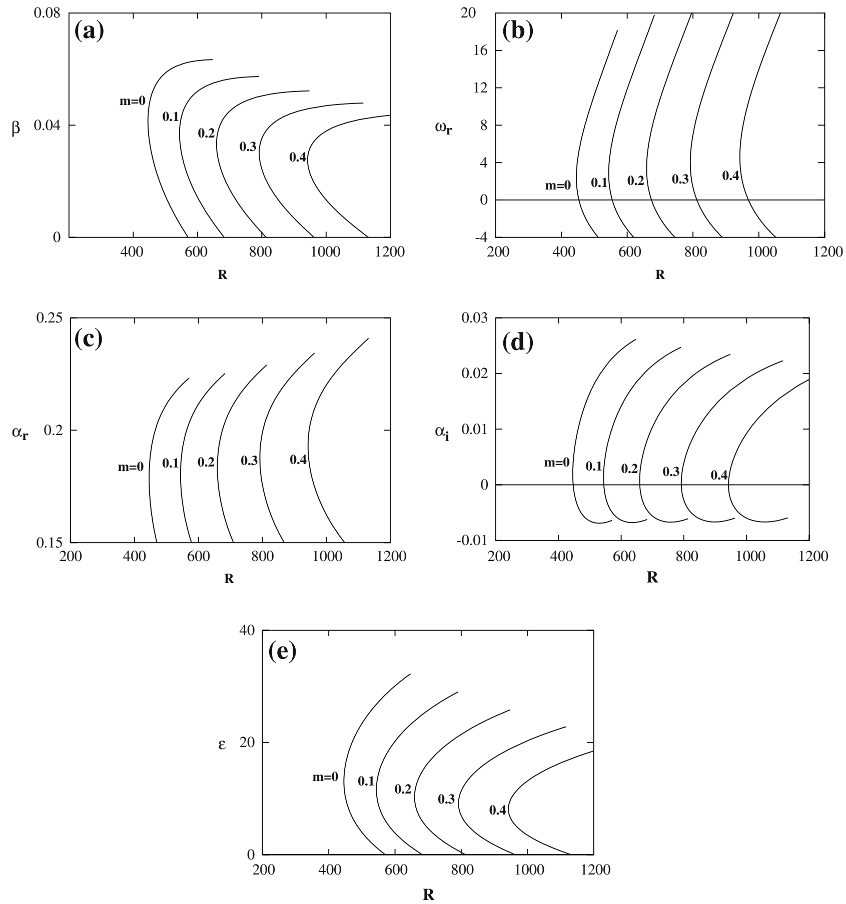

Fig. 2 Direct spatial resonance parameters obtained from the viscous stability equations (11) are shown against the Reynolds number for varying magnetic field-strength parameters, respectively, in (**a**) β, (**b**) ω_r (**c**) α_r, (**d**) α_i and (**e**) ε

straightforward calculation of (14) using the residue theorem produces a time-asymptotic response

$$G \sim \frac{\mathrm{e}^{\mathrm{i}[\alpha(r-r_a)+\beta\theta-\omega t]}}{[2\pi \mathrm{i} t \frac{\partial D}{\partial \omega} \frac{\partial^2 \mathbf{D}}{\partial \alpha^2}]^{1/2}}. \tag{15}$$

We see from (15) that at the resonance point, where the group velocity $\frac{\partial \omega}{\partial \alpha} = 0$ vanishes with a positive ω_i, the Green's function becomes infinite for large enough times pointing to an absolute instability. If, on the other hand, a double pole of (14) occurs, the response can be calculated again using the residue theorem, see Koch [1] for instance, as

$$G \sim \left[(\mathrm{i} A_1 r + A_2)\Psi + A_1 \frac{\partial \Psi}{\partial \alpha} \right] \mathrm{e}^{\mathrm{i}[\alpha(r-r_a)+\beta\theta-\omega t]}, \tag{16}$$

with $A_1,\ A_2$ being functions of the eigenvalues. We observe here that, owing to the term $A_1 r$, a locally algebraic growth is possible, yielding a direct spatial resonance.

We take into account, in the current work, three particular Reynolds numbers leading to resonance, viz. (15) and (16) (and an equation similar to (16) for a direct temporal resonance), since they are of paramount physical significance. The first of these critical Reynolds numbers refers to the onset of the direct spatial resonance point of coalescence coinciding with the neutral stability (that is $\alpha_i = 0$ together with w real). The importance of such a coalescing point was highlighted in [1,20,21] for several boundary-layer

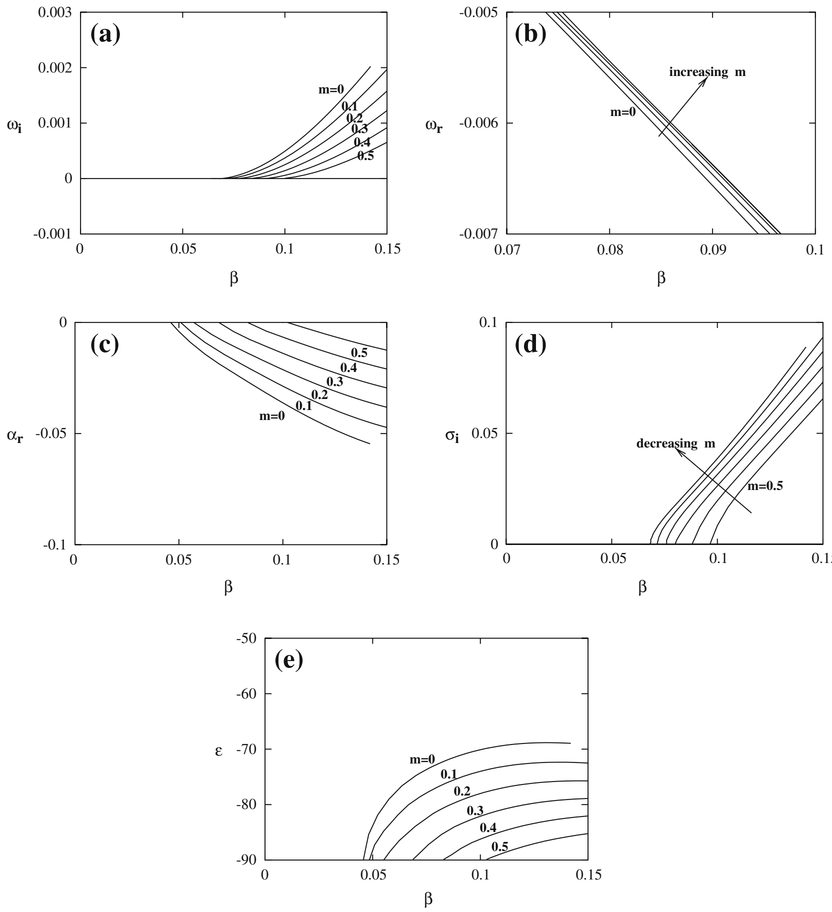

Fig. 3 Direct spatial resonance parameters are shown against the azimuthal wavenumber β in the inviscid limit obtained from the inviscid Rayleigh equation (12) for varying magnetic field-strength parameters respectively in (**a**) ω_i, (**b**) ω_r, (**c**) α_r, (**d**) α_i and (**e**) ε

flows. For the rotating-disk boundary-layer flow, this resonance case as calculated from the linear viscous stability equations (11), and also the inviscid orientation of spatial resonance modes as computed from the inviscid Rayleigh equation (12) are demonstrated in Figs. 2–3. The physical importance implied by the modal coalescence occurring at such critical Reynolds numbers can be better seen from the response given by the Green's function, which asymptotically behaves at large distances downstream like $r\mathrm{e}^{-\alpha_i r}$; see (16). Thus, the local response of the flow is determined by a critical amplitude of a magnitude calculated as the inverse of the spatial amplification rate α_i (only $\alpha_i > 0$ is of concern as far as the direct spatial resonance). When this critical amplitude is denoted by $A = 1/\alpha_i$, it is obvious to state that a large amplitude is attained whenever α_i is close to zero (see particularly Fig. 2d). As a result, this resonance point yields an unbounded algebraic growth (preceding any exponential amplification through other instabilities) which, in turn, is capable of initiating the nonlinear effects and the onset of transition leading to a final breakdown of laminar flow into turbulence. It can also be anticipated that, as compared to the non-magnetic case, the direct spatial resonance occurs at larger Reynolds numbers for increasing magnetic field parameters m; thus, the magnetic field acts to stabilize this resonance mechanism. The inviscid correspondence of the direct spatial resonance parameters, as shown in Fig. 3, reveals the fact that the viscous resonance results displayed in Fig. 2 are not an artifact of the parallel-flow assumption.

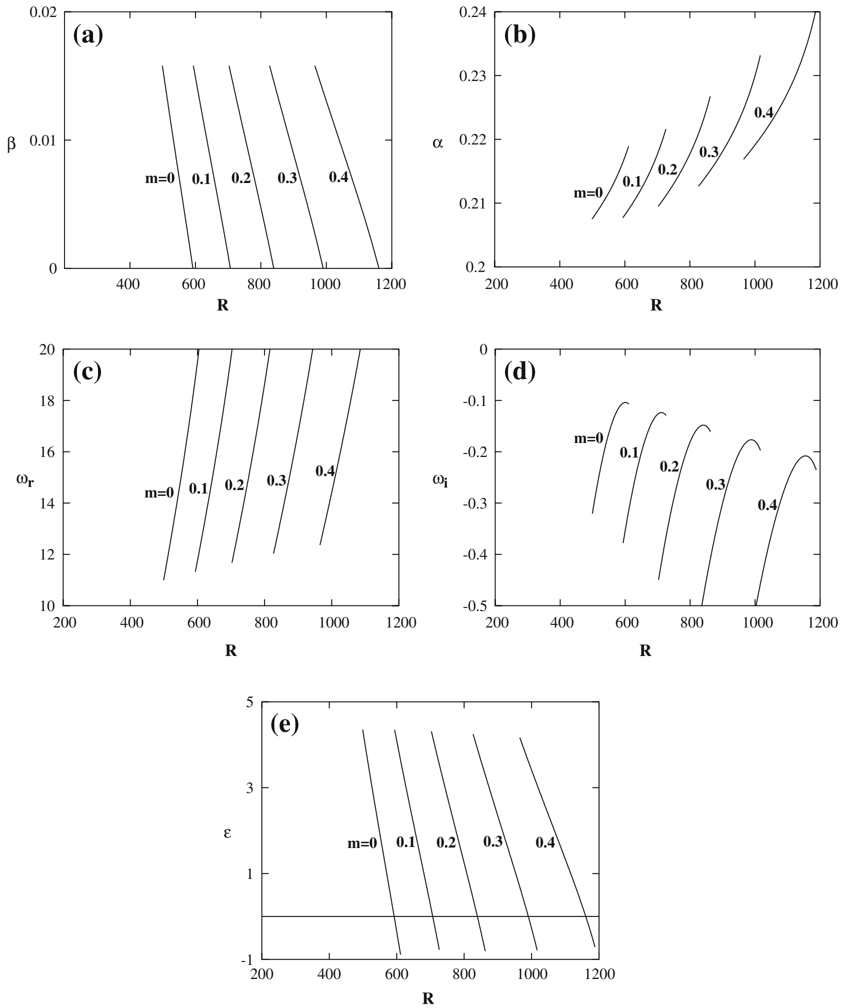

Fig. 4 Direct temporal resonance parameters obtained from the viscous stability equations (11) are shown against the Reynolds number for varying magnetic field-strength parameters, respectively, in (**a**) β, (**b**) α_r (**c**) ω_r, (**d**) ω_i and (**e**) ε

A direct temporal resonance between the merging eigenmodes is also possible in the rotating-disk boundary-layer flow with α real this time, as demonstrated in Fig. 4. Similar to the direct-spatial-resonance case, a resonance Reynolds number near such a point occurs and the local response of the mean flow can be evaluated asymptotically as $te^{\omega_i t}$. Therefore, the critical amplitude of the disturbances at direct temporal resonance conditions is determined by the inverse of the temporal amplification rate ω_i (only $\omega_i < 0$ is of concern for the direct temporal resonance). Whenever $-\omega_i$ is close to zero, the largest amplitude amplification will occur at such resonance Reynolds numbers. Figure 4 (see particularly Fig. 4d) shows that no true temporal resonance is possible with ω_i exactly zero. However, the near-neutral temporal resonance first occurs at higher Reynolds numbers as compared to the spatial resonance Reynolds numbers, as displayed in Fig. 2. On the other hand, the clear implication as deduced from Fig. 4 is that the onset of temporal resonance occurs at much smaller Reynolds numbers. This points to the possible existence of sub-critical nonlinear instabilities, although the modes will be damped in accordance with the linear stability theory. It is also not difficult to observe from Fig. 4 that the stabilizing effect of the presence of a uniform magnetic field applied in the normal direction is felt strongly as the magnetic field parameter increases.

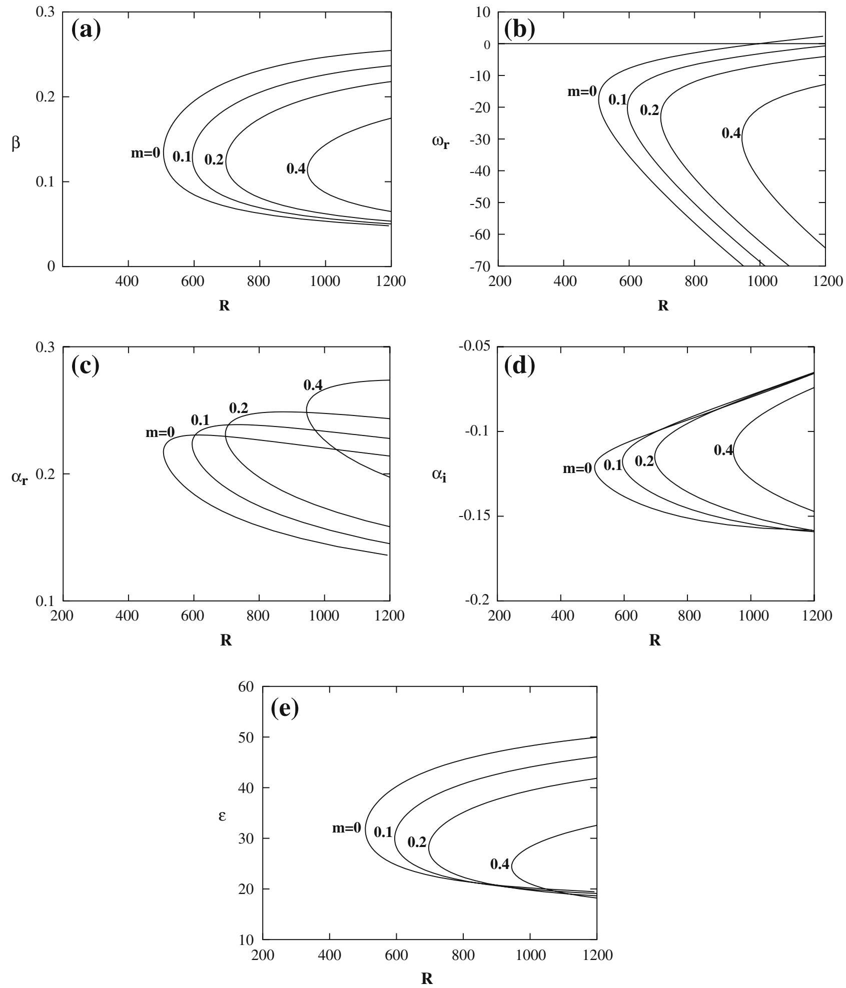

Fig. 5 Viscous absolute instability parameters obtained from the viscous stability equations (11) are shown against the Reynolds number for varying magnetic field-strength values, respectively, in (**a**) β, (**b**) ω_r (**c**) α_r, (**d**) α_i and (**e**) ε

The third resonance Reynolds number occurs as a result of the coalescence of two eigenmodes having zero group velocity. In this case neither of the eigenvalues α or ω need to be real. The only condition is that ω_i be positive to yield absolute growth at resonance. The resonance case leading to viscous absolute instability and also the inviscid orientation of absolute instability are shown in Figs. 5–6. In fact, the local response of the mean flow at such a resonance point is shown to be asymptotically proportional to $e^{\omega_i t}$; see (15). This generates an absolute amplitude amplification at the resonance; see for instance [19, Chapter 3]. An excellent agreement between the viscous absolute instability results of [6] and Fig. 5 can be seen. It should be emphasized here that the present study is not intended to duplicate the results of [6] but, as outlined previously, the absolute instability is a subset of the resonance instability mechanisms under investigation. Here we also computed in Fig. 6 the inviscid modes leading to the absolute instability (which was missing in [6]), whose viscous origin is as shown in Fig. 5. It can be immediately deduced that, at fixed individual magnetic field parameters m, the onset resonance Reynolds number for the absolute instability is larger than that corresponding to the direct spatial resonance instability. Moreover, the same impact of stabilization of the uniform normal magnetic field as happened to the direct spatial/temporal resonances also exists here on the absolute instability mechanism.

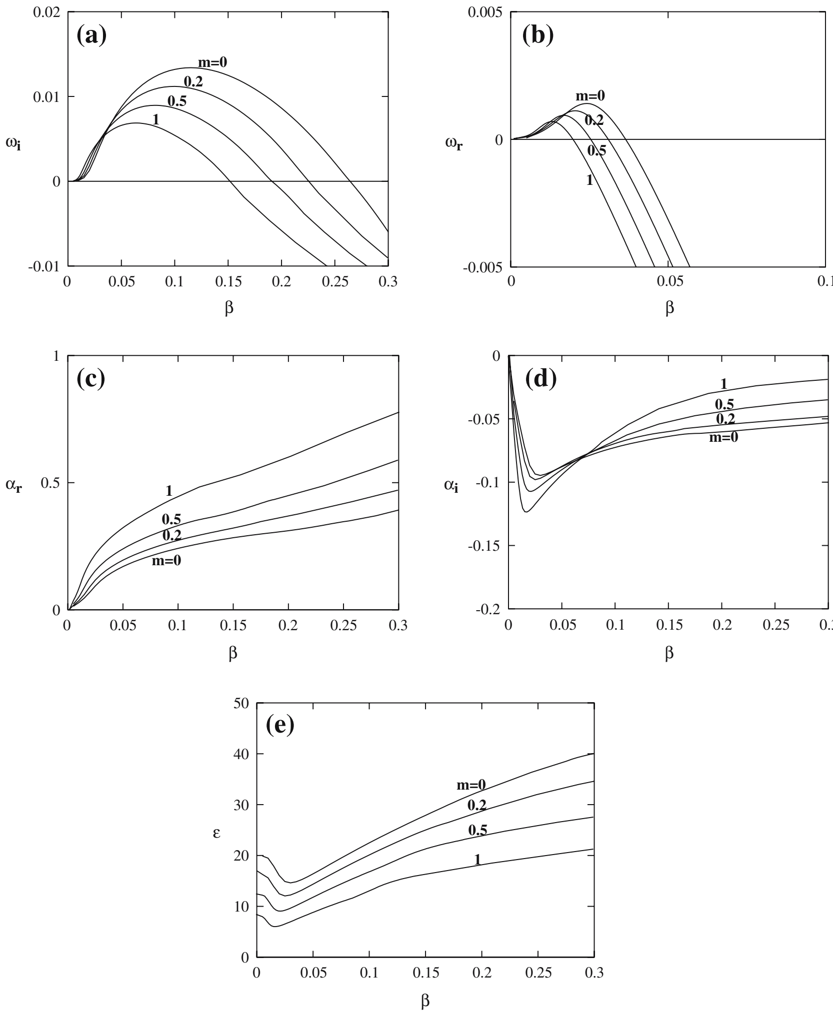

Fig. 6 Inviscid absolute instability parameters obtained from the inviscid Rayleigh equation (12) are shown against β for varying magnetic field-strength values, respectively, in **(a)** ω_i, **(b)** ω_r **(c)** α_r, **(d)** α_i and **(e)** ε

It can thus be concluded from Figs. 2–6 that the magnetic field has a notable stabilizing influence on the resonance instability mechanisms considered, which is expected, owing to the assumption of a low magnetic-field Reynolds number. The effect, however, is shown to be less felt on the direct spatial resonance mechanism, as compared to the other resonance mechanisms. The quantitative indication in the case of a direct spatial resonance mechanism is such that this mechanism is likely to be more important, thus signifying the possibility of a breakdown of laminar flow through the direct spatial resonance instability mechanism. Since from a physical point of view the direct spatial resonance is found to be more relevant, the critical parameters of the direct spatial resonance are depicted in Fig. 7, against the magnetic field-strength parameter m. For comparison purposes, Fig. 7a also plots the critical Reynolds numbers for the absolute instability mechanism by the dashed lines, which gives full support to our lines of argument above.

5 Conclusions

The nature of resonance instability of the viscous and inviscid incompressible boundary-layer flow over a rotating disk has been throughly investigated here, when the flow is subjected to a uniform normal magnetic

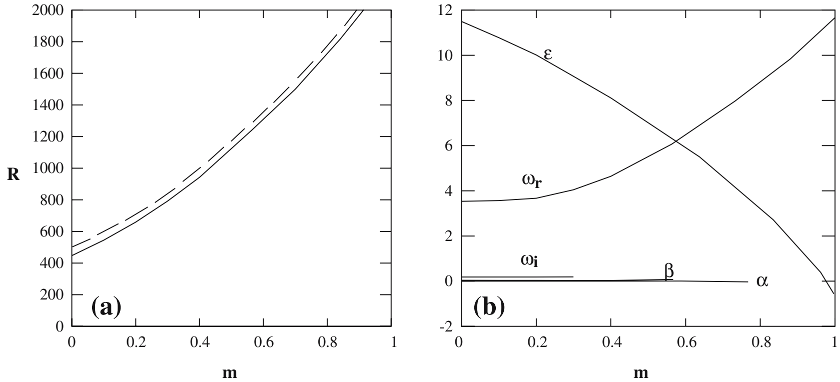

Fig. 7 Critical values are shown for direct spatial resonance instability against the magnetic field-strength parameter *m*, respectively, in (**a**) *R* and (**b**) the other eigenvalues. Part (**a**) also contains the corresponding critical Reynolds numbers to the absolute instability mechanism, indicated by the dashed lines

field. The linearized system of incompressible stability equations was solved numerically by employing a spectral Chebyshev collocation technique, and the coalescing modes to generate, respectively, direct spatial, direct temporal and absolute instability mechanisms have been explored. Comparisons among these resonance mechanisms have been carried out to determine the one which has the most physical relevance, and also, is liable for a possible transition to turbulence in fully three-dimensional boundary-layer flows of practical interest.

The main conclusion of the present research is that, in the presence of a uniform magnetic field within the Von Kármán mean flow, the critical Reynolds number for the onset of resonance instability mechanisms increases sharply, pointing to a strong stabilization effect of the normal magnetic field. This is totally in line with the retarding influence of the low-magnetic-field Reynolds-number approximation on the linear modes. However, for specific strength parameters, the stabilization effect is clearly observed in the case of absolute instability as well as the direct temporal instability, as compared to the direct spatial instability case, even though the possibility of a sub-critical nonlinear instability exists, owing to the temporal resonance. Our results thus enable us to further conclude that, from a physical point of view, the direct spatial resonance instability is a major danger rather than the absolute instability mechanism of Jasmine and Gajjar [6], or than the direct temporal instability mechanism as calculated here. This proves that the direct spatial resonance instability requires extreme care to keep the flow laminar in real-life applications.

Applying a circumferential magnetic field rather than the transverse one, as done in this work, is known to substantially increase the mean flow quantities from the calculations of Pao [22]. Although it can be predicted that in that case the rotating-disk boundary-layer flow will be subjected to more unstable resonance mechanisms, it remains to be justified. This work can also be extended to incorporate the effects of compressibility which requires much further effort.

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